Semantic Cut Elimination for the Logic of Bunched Implications and Structural Extensions

(as formalized in Coq)

Dan Frumin

Grolog, 13 Oct

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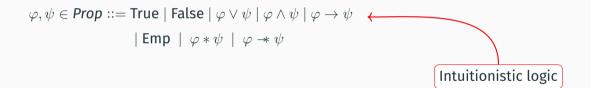
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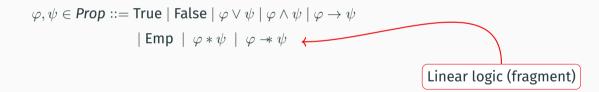
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- Cut elimination: a proof of $\vdash \varphi$ only includes subformulas of φ .
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- Structural extensions: extensions of the logic with certain axioms/rules.
- Formalized in Coq: axiom-free formalization at

https://github.com/co-dan/BI-cutelim.

 $\varphi, \psi \in \mathbf{Prop} ::= \mathsf{True} \mid \mathsf{False} \mid \varphi \lor \psi \mid \varphi \land \psi \mid \varphi \rightarrow \psi$





$$\begin{split} \varphi, \psi \in \textit{Prop} ::= \mathsf{True} \mid \mathsf{False} \mid \varphi \lor \psi \mid \varphi \land \psi \mid \varphi \to \psi \\ \mid \mathsf{Emp} \mid \varphi \ast \psi \mid \varphi \twoheadrightarrow \psi \end{split}$$

$$\begin{array}{ll} \mathsf{True} \land \varphi = \varphi & \varphi \land (\varphi \to \psi) \vdash \psi \\ \mathsf{Emp} \ast \varphi = \varphi & \varphi \ast (\varphi \twoheadrightarrow \psi) \vdash \psi \end{array}$$

$$\begin{array}{ccc} \varphi \wedge \psi \vdash \varphi & \varphi \vdash \varphi \wedge \varphi \\ \varphi \ast \psi \not\vdash \varphi & \varphi \not\vdash \varphi \ast \varphi \end{array}$$

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Proposition represent ownership of resources

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$$\begin{array}{l} \mathsf{Proposition represent ownership of resou} \text{ Both } \varphi \text{ and } \psi \text{ hold for owned resources} \\ \varphi \text{ and } \psi \text{ hold for separate/disjoint resources} \end{array}$$

BI has seen a lot of applications in CS, especially as a basis for *program logics* for programs with arrays/dynamic memory

- $\ell \mapsto v$: the current state has the location ℓ in memory, and it stores the value v
- P * Q: the current state can be divided into two disjoint parts, for which P and Q hold respecively

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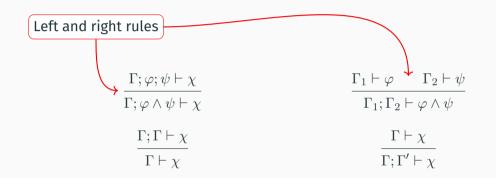
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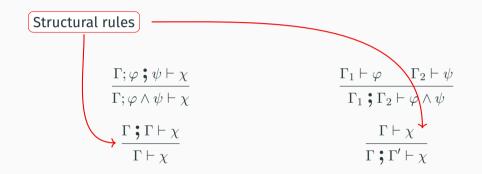
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- $\ell \mapsto v \wedge \ell' \mapsto v'$: aliasing is allowed
- $\ell_1 \mapsto (v_1, \ell_2) * \ell_2 \mapsto (v_2, \ell_3) * \cdots * \ell_n \mapsto (v_n, \ell_o) * \ell_o \mapsto \text{NULL:}$ a linked list without cycles



 $\frac{\Gamma;\varphi;\psi\vdash\chi}{\Gamma;\varphi\wedge\psi\vdash\chi}$

 $\frac{\Gamma_1 \vdash \varphi \quad \Gamma_2 \vdash \psi}{\Gamma_1; \Gamma_2 \vdash \varphi \land \psi}$





Sequent calculus

 $\frac{\Gamma_{1} \vdash \varphi \qquad \Gamma_{2} \vdash \psi}{\Gamma_{1} , \Gamma_{2} \vdash \varphi * \psi}$ $\frac{\Gamma_{1} \vdash \varphi \qquad \Gamma_{2} \vdash \psi}{\Gamma_{1} ; \Gamma_{2} \vdash \varphi \land \psi}$ $\frac{\Gamma \vdash \chi}{\Gamma ; \Gamma' \vdash \chi}$

Sequent calculus

 $\Delta(\varphi, \psi) \vdash \chi$ $\Gamma_1 \vdash \varphi \qquad \Gamma_2 \vdash \psi$ $\Delta(\varphi * \psi) \vdash \chi$ $\Gamma_1 \cdot \Gamma_2 \vdash \varphi * \psi$ $\Delta(\varphi;\psi) \vdash \chi$ $\Gamma_1 \vdash \varphi \qquad \Gamma_2 \vdash \psi$ $\Delta(\varphi \wedge \psi) \vdash \chi$ Γ_1 ; $\Gamma_2 \vdash \varphi \land \psi$ $\Delta(\Gamma; \Gamma) \vdash \chi$ $\Delta(\Gamma) \vdash \chi$ $\overline{\Delta(\Gamma;\Gamma')\vdash \chi}$ $\Delta(\Gamma) \vdash \chi$

$$\left[\Gamma ::= \varphi \mid \Gamma ; \Gamma \mid \Gamma ; \Gamma \mid \dots \right]$$

$$\frac{\Delta \cdot \varphi \vdash \psi}{\Delta \vdash \varphi \twoheadrightarrow \psi} \qquad \qquad \frac{\Delta \cdot \varphi \vdash \psi}{\Delta \vdash \varphi \to \psi}$$

1.

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- Sequent calculus for BI externalizes ∧ and * as different connectives: ; and ,. Only ; admits weakening and contraction.
- Because of that, contexts in the sequents are not lists/multisets, but *trees* (referred to as *bunches*);
- Left rules can be applied deep inside an arbitrary bunched context.

Cut rule

$$\frac{ \overset{\mathsf{CUT}}{\Delta' \vdash \psi} \quad \Delta(\psi) \vdash \varphi}{\Delta(\Delta') \vdash \varphi}$$

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Intuitions:

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- provability relation is transitive

Theorem

Everything that is provable, is also provable without the cut rule: $\vdash \varphi \implies \vdash_{\sf cf} \varphi$

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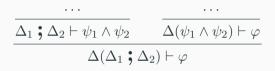
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Why eliminate cut?

- makes the calculus *analytical* (subformula property): any derivation of $\varphi \vdash \psi$ only involves formula that are already present in φ and ψ
- important ingredient in the automated proof search toolbox

Cut elimination

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$$\frac{\Delta_{2} \vdash \psi_{2}}{\Delta(\Delta_{1}; \psi_{2}) \vdash \varphi} \frac{\Delta_{1} \vdash \psi_{1} \quad \Delta(\psi_{1}; \psi_{2}) \vdash \varphi}{\Delta(\Delta_{1}; \psi_{2}) \vdash \varphi}$$

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$$\frac{\Delta_1 ; \Delta_2 \vdash \psi_1 \land \psi_2 \qquad \Delta(\psi_1 \land \psi_2) ; \varphi_1 \vdash \varphi_2}{\Delta(\Delta_1 ; \Delta_2) ; \varphi_1 \vdash \varphi_2} \\
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On the other hand, formalizing these kind of proofs can also be tough...

A semantic proof of cut elimination goes through some "universal" model C and the interpretation of sequent calculus in it.

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$$\mathcal{C}\models\varphi\implies \vdash_{\mathsf{cf}}\varphi$$

BI algebra

A BI algebra (C, \leq) consists of operations $\top, \bot, \lor, \land, \rightarrow, Emp, *, *$ satisfying various laws.

Soundness: $\varphi \vdash \psi \implies \llbracket \varphi \rrbracket \leq \llbracket \psi \rrbracket$.

- $\mathcal{L} = \{ [\varphi] \mid \varphi \in \mathit{Frml} \}$ with $\leq_{\mathcal{L}}$ is a BI algebra;
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- The "real" work is to show that \mathcal{L} is indeed a model.

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What if we use \vdash_{cf} instead of \vdash in the definition of \mathcal{L} ? Need transitivity of $\leq: [\varphi] \leq [\psi] \leq [\chi] \implies [\varphi] \leq [\chi]$? Same as cut elimination: $\varphi \vdash_{cf} \psi \vdash_{cf} \chi \implies \varphi \vdash_{cf} \chi$

Attempted solution: use sets of predecessors.

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$$\langle \varphi \rangle \lor \langle \psi \rangle = \bigcap \{ Y \in \mathcal{C} \mid \langle \varphi \rangle \cup \langle \psi \rangle \subseteq Y \}$$

The smallest set in \mathcal{C} containing $\langle \varphi \rangle \cup \langle \psi \rangle$

Solution: close under arbitrary intersections:

$$\mathcal{C} = \{ \bigcap_{i \in I} \langle \varphi_i \rangle \mid I \text{ arbitrary set}, \varphi_i \in Frml \} \subseteq \wp(Bunch)$$

 $\mathsf{cl}(-): \wp(\mathsf{Bunch}) \to \mathcal{C} \\ \mathsf{cl}(X) = \bigcap \{ \langle \varphi \rangle \mid X \subseteq \langle \varphi \rangle \}$

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Proposition

If $X \in \mathcal{C}$ with $\Delta_1, \ldots, \Delta_n \in X$ and

$$\frac{\Delta_1 \vdash \varphi \quad \dots \quad \Delta_n \vdash \varphi}{\Delta' \vdash \varphi}$$

without the use of the cut rule, then $\Delta' \in X$.

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Lift operations to *C*:

$$\begin{split} X \wedge Y &= X \cap Y \\ X \vee Y &= \mathsf{cl}(X \cup Y) \\ X * Y &= \mathsf{cl}(\{\Delta_1 \ \ \mathbf{0} \ \Delta_2 \mid \Delta_1 \in X, \Delta_2 \in Y\}) \\ \end{split} \qquad \begin{array}{l} X \to Y &= \{\Delta \mid \forall \Delta' \in X. \ (\Delta \ \mathbf{0} \ \mathbf{0} \ \Delta') \in Y\} \\ X - * Y &= \{\Delta \mid \forall \Delta' \in X. \ (\Delta \ \mathbf{0} \ \mathbf{0} \ \Delta') \in Y\} \\ \end{array} \end{split}$$

Proposition

 $\ensuremath{\mathcal{C}}$ is a BI algebra

Fundamental property

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Cut elimination

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If $\varphi \vdash \psi$, then $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$.

 $\mathsf{If}\,\llbracket\varphi\rrbracket \subseteq \llbracket\psi\rrbracket, \mathsf{then}\; \varphi \in \llbracket\varphi\rrbracket \subseteq \llbracket\psi\rrbracket \subseteq \langle\psi\rangle \implies \varphi \vdash_\mathsf{cf} \psi.$

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• BI + additional structural rules, e.g. affine BI

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• BI + \Box modality: BI based on IS4

$$\begin{array}{c} \Box \mathsf{L} & & \Box \mathsf{R} \\ \underline{\Delta(A) \vdash B} & & \underline{\Box \Delta \vdash A} \\ \hline \Delta(\Box A) \vdash B & & \overline{\Box \Delta \vdash \Box A} \end{array}$$

 $\ensuremath{\mathcal{C}}$ is a BI algebra with a modal operator

An analytic structural rule is of the form

$$\frac{\Pi(T_1[\Delta_1,\ldots,\Delta_n])\vdash\varphi\quad\ldots\quad\Pi(T_m[\Delta_1,\ldots,\Delta_n])\vdash\varphi}{\Pi(T[\Delta_1,\ldots,\Delta_n])\vdash\varphi}$$

where T_1, \ldots, T_m, T are bunched terms – bunches built out of connectives , ;, and variables x_1, \ldots, x_n , and T is linear

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Corresponds to the axiom:

$$\llbracket T \rrbracket(p_1,\ldots,p_n) \leq \llbracket T_1 \rrbracket(p_1,\ldots,p_n) \vee \cdots \vee \llbracket T_m \rrbracket(p_1,\ldots,p_n).$$

How do we verify that

$$[[T]](p_1,\ldots,p_n) \leq [[T_1]](p_1,\ldots,p_n) \vee \cdots \vee [[T_m]](p_1,\ldots,p_n).$$

holds in the model C?

Proposition

For $X_1, X_2, \ldots, X_n \in \mathcal{C}$,

 $\mathsf{cl}(\{T[\Delta_1,\ldots,\Delta_n] \mid \Delta_i \in X_i, 1 \le i \le n\}) \subseteq \llbracket T \rrbracket(X_1,\ldots,X_n).$

And this becomes an equality when \boldsymbol{T} is linear.

$$\frac{\Pi(T_1[\Delta_1,\ldots,\Delta_n])\vdash\varphi\quad\ldots\quad\Pi(T_m[\Delta_1,\ldots,\Delta_n])\vdash\varphi}{\Pi(T[\Delta_1,\ldots,\Delta_n])\vdash\varphi}$$

What if T is not linear?

$$\frac{\Pi(T_1[\Delta_1,\ldots,\Delta_n])\vdash\varphi\quad\ldots\quad\Pi(T_m[\Delta_1,\ldots,\Delta_n])\vdash\varphi}{\Pi(T[\Delta_1,\ldots,\Delta_n])\vdash\varphi}$$

What if *T* is not linear?

Then we can turn the above rule into an equivalent analytic rule.

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 \Rightarrow

 $\Pi(\Delta_1, \Delta_2) \vdash \varphi$

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corresponds to $p * p \le p$

 $\begin{array}{c} \Rightarrow \\ \\ \frac{\Pi(\Delta_1) \vdash \varphi \quad \Pi(\Delta_2) \vdash \varphi}{\Pi(\Delta_1 \ {\color{red}{\bullet}} \ \Delta_2) \vdash \varphi} \end{array} \\ \end{array} \\$

corresponds to $p_1 * p_2 \le p_1 \lor p_2$

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Clearly $p_1 * p_2 \le p_1 \lor p_2$ implies $p * p \le p$. For the other way around:

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- Extensive use of setoids and setoid rewriting, based on the typeclasses from the stdpp library
- Turn equations $\Delta = \Delta'(\Gamma)$ into inductive systems Inductive bunch_decomp : bunch \rightarrow bunch_ctx \rightarrow bunch \rightarrow Prop

Thank you for your attention!

Let me know if you have questions, d.frumin@rug.nl.