Semantic Cut Elimination
for the Logic of Bunched Implications
(as formalized in Coq)

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Semantic cut elimination for the logic of Bunched Implications, formalized in Coq.
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- **Cut elimination**: a proof of $\Gamma \vdash \varphi$ only includes subformulas of $\varphi$.
- **Semantic proof**: proof by interpreting syntax in a model.
- **Formalized in Coq**: axiom-free formalization at
  
The logic of Bunched Implications

BI freely combines intuitionistic and linear connectives:

\[ \varphi, \psi \in \text{Prop} ::= \text{True} \mid \text{False} \mid \varphi \lor \psi \mid \varphi \land \psi \mid \varphi \rightarrow \psi \]
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| Emp | \varphi \ast \psi | \varphi \rightarrow \ast \psi |

Intuitionistic logic
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| \text{Emp} | \varphi \ast \psi | \varphi \rightarrow \ast \psi \]

Linear logic (fragment)
The logic of Bunched Implications

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$$\phi, \psi \in Prop ::= \text{True} \mid \text{False} \mid \phi \lor \psi \mid \phi \land \psi \mid \phi \rightarrow \psi$$

$$\mid \text{Emp} \mid \phi \ast \psi \mid \phi \ast \psi$$

Proposition represent ownership of resources
Sequent calculus

Sequent: $\Gamma \vdash \phi$

$\frac{\Gamma; \varphi; \psi \vdash \chi}{\Gamma; \varphi \land \psi \vdash \chi}$

$\frac{\Gamma_1 \vdash \varphi \quad \Gamma_2 \vdash \psi}{\Gamma_1; \Gamma_2 \vdash \varphi \land \psi}$
Sequent calculus

Left and right rules

\[
\frac{\Gamma; \varphi; \psi \vdash \chi}{\Gamma; \varphi \wedge \psi \vdash \chi}
\]

\[
\frac{\Gamma; \Gamma \vdash \chi}{\Gamma \vdash \chi}
\]

\[
\frac{\Gamma_1 \vdash \varphi}{\Gamma_1; \Gamma_2 \vdash \varphi \wedge \psi}
\]

\[
\frac{\Gamma_2 \vdash \psi}{\Gamma_1; \Gamma_2 \vdash \varphi \wedge \psi}
\]

\[
\frac{\Gamma \vdash \chi}{\Gamma; \Gamma' \vdash \chi}
\]
Sequent calculus

Structural rules

\[ \Gamma; \varphi \land \psi \vdash \chi \]

\[ \frac{\Gamma; \varphi \vdash \chi, \Gamma; \psi \vdash \chi}{\Gamma; \varphi \land \psi \vdash \chi} \]

\[ \frac{\Gamma; \varphi \vdash \chi, \Gamma; \psi \vdash \chi}{\Gamma \vdash \chi} \]

\[ \frac{\Gamma_1 \vdash \varphi, \Gamma_2 \vdash \psi}{\Gamma_1; \Gamma_2 \vdash \varphi \land \psi} \]

\[ \frac{\Gamma \vdash \chi}{\Gamma; \Gamma' \vdash \chi} \]
Sequent calculus

\[
\frac{\Gamma; \varphi, \psi \vdash \chi}{\Gamma; \varphi * \psi \vdash \chi}
\]

\[
\frac{\Gamma; \varphi ; \psi \vdash \chi}{\Gamma; \varphi \land \psi \vdash \chi}
\]

\[
\frac{\Gamma; \Gamma \vdash \chi}{\Gamma \vdash \chi}
\]

\[
\frac{\Gamma_1 \vdash \varphi \quad \Gamma_2 \vdash \psi}{\Gamma_1, \Gamma_2 \vdash \varphi \land \psi}
\]

\[
\frac{\Gamma_1 ; \Gamma_2 \vdash \varphi \land \psi}{\Gamma_1 ; \Gamma_2 \vdash \varphi \land \psi}
\]

\[
\frac{\Gamma \vdash \chi}{\Gamma ; \Gamma' \vdash \chi}
\]
Sequent calculus

\[
\begin{align*}
\Delta(\varphi, \psi) & \vdash \chi \\
\Delta(\varphi \ast \psi) & \vdash \chi \\
\Delta(\varphi ; \psi) & \vdash \chi \\
\Delta(\varphi \land \psi) & \vdash \chi \\
\Delta(\Gamma ; \Gamma) & \vdash \chi \\
\Delta(\Gamma) & \vdash \chi
\end{align*}
\]

\[
\begin{align*}
\Gamma_1 \vdash \varphi & \quad \Gamma_2 \vdash \psi \\
\Gamma_1 \ast \Gamma_2 & \vdash \varphi \land \psi \\
\Gamma_1 ; \Gamma_2 & \vdash \varphi \land \psi \\
\Delta(\Gamma) & \vdash \chi \\
\Delta(\Gamma ; \Gamma') & \vdash \chi
\end{align*}
\]

\[
\Gamma ::= \varphi \mid \Gamma ; \Gamma \mid \Gamma , \Gamma \mid \ldots
\]
Cut rule

\[
\text{CUT} \quad \quad \Delta' \vdash \psi \quad \Delta(\psi) \vdash \varphi \\
\hline
\Delta(\Delta') \vdash \varphi
\]
Intuitions:

- $\psi$ is an “intermediate lemma”
**Cut rule**

\[
\text{CUT} \\
\Delta' \vdash \psi \\
\Delta(\psi) \vdash \varphi \\
\hline
\Delta(\Delta') \vdash \varphi
\]

**Intuitions:**

- \( \psi \) is an “intermediate lemma”
- provability relation is transitive
Theorem

Everything that is provable, is also provable without the cut rule: \( \vdash \varphi \iff \vdash_{cf} \varphi \)
Cut elimination

**Theorem**

Everything that is provable, is also provable without the cut rule: $\vdash \varphi \iff \vdash_{\text{cf}} \varphi$

Why eliminate cut?

- makes the calculus *analytical* (subformula property): any derivation of $\varphi \vdash \psi$ only involves formula that are already present in $\varphi$ and $\psi$
- important ingredient in the automated proof search toolbox
Usually proofs of cut elimination involve analysis by inversion + terminating measure:

\[ \Delta_1 \vdash \psi_1 \land \psi_2 \quad \Delta(\psi_1 \land \psi_2) \vdash \varphi \]

\[ \Delta(\Delta_1 ; \Delta_2) \vdash \varphi \]
Usually proofs of cut elimination involve analysis by inversion + terminating measure:

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\Delta_1 \triangledown \Delta_2 \vdash \psi_1 \land \psi_2 \\
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\Delta(\Delta_1 \triangledown \Delta_2) \vdash \varphi
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Usually proofs of cut elimination involve analysis by inversion + terminating measure:

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\Delta_1 \vdash \psi_1 & \quad \Delta_2 \vdash \psi_2 \\
\Delta_1 ; \Delta_2 \vdash \psi_1 \land \psi_2 & \quad \Delta(\psi_1 ; \psi_2) \vdash \varphi \\
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\Delta_1 \vdash \psi_1 & \quad \Delta_2 \vdash \psi_2 \quad \Delta(\psi_1 ; \psi_2) \vdash \varphi \\
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\psi_1 & \quad \Delta_2 \vdash \psi_2 \\
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\Delta(\Delta_1 \triangleright \Delta_2) & \vdash \varphi
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\Delta_1 & \vdash \psi_1 & \Delta_2 & \vdash \psi_2 \\
\Delta_1 ; \Delta_2 & \vdash \psi_1 \land \psi_2 \\
\Delta & \vdash (\psi_1 \land \psi_2) ; \varphi_1 \vdash \varphi_2 \\
\Delta & \vdash (\Delta_1 ; \Delta_2) \vdash \varphi_1 \rightarrow \varphi_2
\end{align*}
\]
Cut elimination

Usually proofs of cut elimination involve analysis by inversion + terminating measure:

\[
\frac{\Delta_1 \vdash \psi_1 \quad \Delta_2 \vdash \psi_2}{\Delta_1 \& \Delta_2 \vdash \psi_1 \land \psi_2} \quad \frac{\Delta(\psi_1 \land \psi_2) \; \varphi_1 \vdash \varphi_2}{\Delta(\psi_1 \land \psi_2) \vdash \varphi_1 \to \varphi_2}
\]

\[
\frac{\Delta(\Delta_1 \& \Delta_2) \vdash \varphi_1 \to \varphi_2}{\Delta(\Delta_1 \& \Delta_2) \vdash \varphi_1 \to \varphi_2}
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Cut elimination

Usually proofs of cut elimination involve analysis by inversion + terminating measure:

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\begin{align*}
\Delta_1 \vdash \psi_1 \land \psi_2 & \quad \Delta_2 \vdash \psi_1 \land \psi_2 \\
\Delta(\psi_1 \land \psi_2) & \vdash \varphi
\end{align*}
\]

\[
\Delta(\Delta_1, \Delta_2) \vdash \varphi
\]

\[\Rightarrow\text{ etc.}\]
Limitations of the direct-style proof

• There are a lot of cases to consider, with a lot of syntactic details
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For these reason, non-formalized proofs of cut elimination can be fragile and are known to be error-prone.
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- Well-foundedness/termination measures can get complicated
- BI specific: the tree-like structure of bunches contribute to the complexity

For these reason, non-formalized proofs of cut elimination can be fragile and are known to be error-prone.

On the other hand, formalizing these kind of proofs can also be tough...
A semantic proof of cut elimination goes through some “universal” model $C$ and the interpretation of sequent calculus in it.

\[ C \models \varphi \implies \vdash_{cf} \varphi \]
A semantic proof of cut elimination goes through some “universal” model $C$ and the interpretation of sequent calculus in it.

$$C \models \varphi \implies \Gamma_{cf} \varphi$$

**BI algebra**

A BI algebra $(C, \leq)$ consists of operations $\top$, $\bot$, $\lor$, $\land$, $\rightarrow$, Emp, $\ast$, $\ast\ast$ satisfying various laws.

**Soundness:** $\varphi \vdash \psi \implies \llbracket \varphi \rrbracket \leq \llbracket \psi \rrbracket$. 
Define $[\varphi] = \{\psi \mid \varphi \vdash \psi\}$, and $[\varphi] \leq_L [\psi] \iff \varphi \vdash \psi$. 

• $L = \{[\varphi] \mid \varphi \in \text{Frml}\}$ with $\leq_L$ is a BI algebra; 

• Main property of $L$: $J\varphi^K = [\varphi]$. 

• Completeness: suppose $\varphi \dashv \vdash \psi$. 

• In particular: $J\varphi^K \leq_L J\psi^K$, i.e. $[\varphi] \leq_L [\psi]$; 

• Conclusion: $\varphi \vdash \psi$. 

• The “real” work is to show that $L$ is indeed a model.
Define $[\varphi] = \{\psi \mid \varphi \vdash \psi\}$, and $[\varphi] \leq_{\mathcal{L}} [\psi] \iff \varphi \vdash \psi$.

- $\mathcal{L} = \{[\varphi] \mid \varphi \in \text{Frml}\}$ with $\leq_{\mathcal{L}}$ is a BI algebra;
- Main property of $\mathcal{L}$: $[\varphi] = [\varphi]$. 
Define \([\varphi] = \{\psi \mid \varphi \vdash \psi\}\), and \([\varphi] \leq_L [\psi] \iff \varphi \vdash \psi\).

- \(\mathcal{L} = \{[\varphi] \mid \varphi \in Frml\}\) with \(\leq_L\) is a BI algebra;
- Main property of \(\mathcal{L}\): \([\varphi] = [\varphi]\);
- Completeness: suppose \(\varphi \models \psi\).
Define $[\varphi] = \{\psi \mid \varphi \vdash \psi\}$, and $[\varphi] \leq_{L} [\psi] \iff \varphi \vdash \psi$.

- $L = \{[\varphi] \mid \varphi \in Frml\}$ with $\leq_{L}$ is a BI algebra;
- Completeness: suppose $\varphi \models \psi$.
  - In particular: $[\varphi] \leq_{L} [\psi]$, i.e. $[\varphi] \leq_{L} [\psi]$;

Conclusion: $\varphi \vdash \psi$. The "real" work is to show that $L$ is indeed a model.
Intuition: Lindenbaum-Tarski algebra for completeness

Define \([\varphi] = \{\psi \mid \varphi \vdash \psi\}\), and \([\varphi] \leq L [\psi] \iff \varphi \vdash \psi\).

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- Main property of \(L\): \([\varphi] = [\varphi]\).
- Completeness: suppose \(\varphi \models \psi\).
  - In particular: \([\varphi] \leq_L [\psi]\), i.e. \([\varphi] \leq_L [\psi]\);
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Define $[\varphi] = \{\psi \mid \varphi \vdash \psi\}$, and $[\varphi] \leq_L [\psi] \iff \varphi \vdash \psi$.

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  - In particular: $[\varphi] \leq_L [\psi]$, i.e. $[\varphi] \leq_L [\psi]$;
  - Conclusion: $\varphi \vdash \psi$.
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What if we use $\vdash_{cf}$ instead of $\vdash$ in the definition of $L$?
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Need transitivity of $\leq$: $[\varphi] \leq [\psi] \leq [\chi] \implies [\varphi] \leq [\chi]$?
What if we use $\vdash_{cf}$ instead of $\vdash$ in the definition of $\mathcal{L}$?

Need transitivity of $\leq$: $[\varphi] \leq [\psi] \leq [\chi] \implies [\varphi] \leq [\chi]$?

Same as cut elimination: $\varphi \vdash_{cf} \psi \vdash_{cf} \chi \implies \varphi \vdash_{cf} \chi$
Attempted solution: use sets of predecessors.

\[ \langle \varphi \rangle = \{ \Delta \mid \Delta \vdash_{\text{cf}} \varphi \} \in \wp(\text{Bunch}), \]

with the subset inclusion relation.
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with the subset inclusion relation.

Note that \( \varphi \in \langle \varphi \rangle \). Hence, \( \langle \varphi \rangle \subseteq \langle \psi \rangle \) implies

\[ \varphi \in \langle \psi \rangle \iff \varphi \vdash_{\text{cf}} \psi. \]
Attempted solution: use sets of predecessors.

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\[ \varphi \in \langle \psi \rangle \iff \varphi \vdash_{\text{cf}} \psi. \]
Is \( \langle \varphi \rangle | \varphi \in Frml \), \( \subseteq \) a BI algebra?
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Not closed under \( \cup, \cap \ldots \) Cannot inherit the algebra structure from \( \varphi(Bunch) \).
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Not closed under \( \cup, \cap \ldots \) Cannot inherit the algebra structure from \( \varphi(Bunch) \).

Solution: close under arbitrary intersections:

\[
C = \left\{ \bigcap_{i \in I} \langle \varphi_i \rangle \mid I \text{ arbitrary set, } \varphi_i \in Frml \right\} \subseteq \varphi(Bunch)
\]
Is \( \{ \langle \varphi \rangle \mid \varphi \in Frml \}, \subseteq \) a BI algebra?

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\[
\text{cl}(\cdot) : \wp(Bunch) \rightarrow C
\]

\[
\text{cl}(X) = \bigcap \{ \langle \varphi \rangle \mid X \subseteq \langle \varphi \rangle \}
\]
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\]

The smallest set in \( C \) containing \( X \)

\[
\text{cl}(\_): \varphi(Bunch) \rightarrow C
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Solution: close under arbitrary intersections

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C = \bigcap_{i \in I} \langle \varphi_i \rangle \mid I \text{ arbitrary}
\]

Lift operations to \( C \):

\[
\begin{align*}
X \land Y &= X \cap Y \\
X \lor Y &= \text{cl}(X \cup Y) \\
X \ast Y &= \text{cl}(\{\Delta_1, \Delta_2 \mid \Delta_1 \in X, \Delta_2 \in Y\})
\end{align*}
\]

\( \text{cl}(-) : \wp(\text{Bunch}) \rightarrow C \)

\( \text{cl}(X) = \bigcap\{\langle \varphi \rangle \mid X \subseteq \langle \varphi \rangle\} \)
Attempt 3

Is \( \{\langle \varphi \rangle \mid \varphi \in Frml\}, \subseteq \) a BI algebra?

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\end{align*}
\]

\( \text{cl}(\cdot) : \varphi(Bunch) \rightarrow C \)

\( \text{cl}(X) = \bigcap \{\langle \varphi \rangle \mid X \subseteq \langle \varphi \rangle\} \)

\( \text{Satisfies} \ [\varphi] \subseteq [\psi] \implies \varphi \vdash \text{cf} \ \psi \)
• Semantic proof of cut elimination through $C$
Sum up

• Semantic proof of cut elimination through $C$
• More modular proof
• Semantic proof of cut elimination through $C$
• More modular proof
• Extensions: structural rules, □ modality.
Reflecting on the formalization

Coq formalization, ~650 lines specs and ~2500 lines proof
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- Good representation for $C$ makes life easier

```coq
Record C := { 
  CPred :> Bunch → Prop;
  CClosed : .... }
```
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• Good representation for $C$ makes life easier

\[
\text{Record } C := \{ \\
\quad \text{CPred :> Bunch} \to \text{Prop}; \\
\quad \text{CClosed} : \ldots \}
\]

• Setoids and setoid rewriting were helpful, useful type classes in \texttt{stdpp}
Reflecting on the formalization

Coq formalization, ~650 lines specs and ~2500 lines proof

• Good representation for \( C \) makes life easier

\[
\text{Record } C := \{ \\
\text{CPred} : \to \text{Bunch} \to \text{Prop}; \\
\text{CClosed} : \ldots \}
\]

• Setoids and setoid rewriting were helpful, useful type classes in \texttt{stdpp}

• Turn equations \( \Delta = \Delta'(\Gamma) \) into inductive systems

\[
\text{Inductive } \text{bunch_decomp} : \text{bunch} \to \text{bunch_ctx} \to \text{bunch} \to \text{Prop}
\]
Thank you for listening!

Let me know if you have questions, d.frumin@rug.nl.