Propositions as Sessions

Logical Foundations of Concurrent Computation

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This Course

A bird's eye view on the logical foundations of concurrent computation.

Plan:

1. Motivation (Jorge) - Multiplicative Linear Logic (MLL) (Dan)
2. The concurrent interpretation of MLL (Jorge)
3. Cut-elimination and correctness for concurrent processes (Jorge)
4. **Beyond linear resources: the !-modality and resource sharing** (Dan)
5. An alternative view of resource sharing: Bunched Implications (Dan)

Propositions As Sessions

Linear logic propositions	$\leftrightarrow$	types describing behavior (sessions)
Sequence calculus derivations	$\leftrightarrow$	π-calculus processes
Cut reductions	$\leftrightarrow$	communication between processes

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Today: incorporating *servers* into the framework.

Session Types

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- Processes can send/receive messages on names, using sequencing and parallel composition. We shall gradually "extract" their syntax from proofs.

Session Types

- Key idea: Interpret the logical sequent

$$A_1, \cdots, A_n \vdash C$$

as a *typing judgment*, under a suitable reading of propositions as sessions:

$$x_1 : A_1, \cdots, x_n : A_n \vdash P :: z : C$$

Process P offers session C on channel z...

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as a *typing judgment*, under a suitable reading of propositions as sessions:

$$x_1 : A_1, \cdots, x_n : A_n \vdash P :: z : C$$

Process P offers session C on channel z...

... by relying on sessions $A_1, \ldots, A_n$ on channels $x_1, \ldots, x_n$

Propositions as Session Types: Cut

P can provide A along x

Q relies on x along A to provide $z : C$

$$\frac{\Delta_1 \vdash P :: x : A \qquad \Delta_2, x : A \vdash Q :: z : C}{\Delta_1, \Delta_2 \vdash (\nu x)(P \mid Q) :: z : C}$$

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Execute *P* and *Q* in parallel

Declare channel *x* **local** to *P* and *Q*

Reductions of processes

$$(\nu x)\big((\nu y)\overline{x}\langle y\rangle.(P_1 \mid P_2) \mid x(z).Q\big) \longrightarrow (\nu x)\big(P_2 \mid (\nu y)(P_1 \mid Q\{^{y}/_{z}\})\big)$$

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$$\frac{P \longrightarrow P'}{(\nu x)(P \mid Q) \longrightarrow (\nu x)(P' \mid Q)} \qquad \frac{Q \longrightarrow Q'}{(\nu x)(P \mid Q) \longrightarrow (\nu x)(P \mid Q')}$$

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$$\frac{P \equiv R \qquad R \longrightarrow R' \qquad R' \equiv P'}{P \longrightarrow P'}$$

A bit of notation

We write $\overline{x}[y].P$ for $(\nu y)\overline{x}\langle y\rangle.P$, e.g.

$$(\nu x)\big(\overline{x}[y].(P_1 \mid P_2) \mid x(z).Q\big) \longrightarrow (\nu x)\big(P_2 \mid (\nu y)(P_1 \mid Q\{y/z\})\big)$$

! Modality in Linear Logic

Linear Logic Menu

Summer Menu! €15 p.p.
Starter: Greek Salad, or Soup
Main: Chicken Schnitzel, Tofu, or Salmon (for additional €2)
(all main dishes come with a side of fries)
Desert: Ice Cream, or Cheese Platter
Drinks: Leffe Tripel (€3 for a glass)

$$E^{15} \multimap \begin{pmatrix} (\mathit{Sal} \,\&\, \mathit{Soup}) \otimes \\ ((\mathit{Schn} \,\&\, \mathit{Tofu} \,\&\, (E \otimes E \multimap \mathit{Fish})) \\ \otimes \;\; \mathit{Fries}) \otimes \\ (\mathit{Icecr} \,\&\, \mathit{Cheese}) \otimes \\ \\ (E^3 \multimap \mathit{Beer}) \end{pmatrix}$$

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Restricted vs unrestricted resources

Intuitionistic (and classical) logic:

$$A \wedge (A \rightarrow B) \vdash B \wedge A \wedge (A \rightarrow B)$$

Linear Logic:

$$A \otimes (A \multimap B) \vdash B$$
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The ! Modality

$!A$ *of course* A, or *bang* A, or *exponential* A

$$!A \vdash A \qquad !A \vdash !A \otimes !A \qquad !A \vdash \mathbf{1} \qquad \frac{A \vdash B}{!A \vdash !B}$$

- $!A$ satisfies contraction, weakening

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- $A \otimes !(A \multimap B) \vdash B \otimes !(A \multimap B)$

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Some useful properties of !:

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- $!(A \mathbin{\&} B) \vdash\ !A \mathbin{\&}\ !B$
- $!A \mathbin{\&}\ !B \nvdash\ !(A \mathbin{\&} B)$

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- $!A \,\&\, !B \nvdash\, !(A \,\&\, B)$
- $!A \dashv\vdash\, !!A$

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- $!A$ - unlimited copies of the session A / server for session A

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- We have two kinds of contexts: $\Gamma; \Delta \vdash C$
  - DILL - Dual Intuitionistic Linear Logic
  - Unrestricted resources in Γ, restricted resources in Δ
  - $A_1, \ldots, A_k; B_1, \ldots, B_n \vdash C$ stands for $!A_1 \otimes \ldots \otimes !A_k \otimes B_1 \otimes \ldots \otimes B_n \multimap C$

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- Unrestricted resources are non-linear and can be shared:

$$\frac{\Gamma; \Delta_1 \vdash P :: x : A \qquad \Gamma; x : A, \Delta_2 \vdash Q :: z : C}{\Gamma; \Delta_1, \Delta_2 \vdash (\nu x)(P \mid Q) :: z : C}$$

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$$\frac{\Gamma;\ \Delta_1 \vdash P :: y : A \qquad \Gamma; \Delta_2 \vdash Q :: x : B}{\Gamma;\ \Delta_1, \Delta_2 \vdash \overline{x}[y].(P \mid Q) :: x : A \otimes B}$$

$$\Gamma;\ x : A \vdash [y \leftarrow x] :: y :: A$$

...etc

The ! Modality as a Session Type

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...*etc*

$$\frac{x : A, y : A, \Gamma;\ \Delta \vdash P :: z : C}{x : A, \Gamma;\ \Delta \vdash P\{x/y\} :: z : C}$$

$$\frac{\Gamma;\ \Delta \vdash P :: z : C}{x : A, \Gamma;\ \Delta \vdash P :: z : C}$$

The ! Modality: Servers

$$\frac{\Gamma; \emptyset \vdash \qquad A}{\Gamma; \emptyset \vdash \qquad !A}$$

- Can derive $!A$ from A, if the proof of A does **not** depend on restricted resources

The ! Modality: Servers

$$\frac{\Gamma; \emptyset \vdash P :: y : A}{\Gamma; \emptyset \vdash \; !u(y).P :: u : !A}$$

- Can derive $!A$ from A, if the proof of A does **not** depend on restricted resources
- $!u(y).P$ receives a request (with a name) on u, and provide P for that request

The ! Modality: Clients

$$\frac{\Gamma; x : !A, \Delta \vdash \quad C}{u : A, \Gamma; \Delta \vdash \quad C} \qquad \frac{u : A, \Gamma; y : A, \Delta \vdash \quad C}{u : A, \Gamma; \Delta \vdash \quad C}$$

The ! Modality: Clients

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- Can move $!A$ from the restricted to the unrestricted section

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- Can move $!A$ from the restricted to the unrestricted section
- Client requests a copy of A by creating a new name and sending the request to $!A$

The ! Modality: Client-Server interaction

$$\dfrac{\dfrac{\Gamma; \emptyset \vdash P :: x : A}{\Gamma; \emptyset \vdash !u(x).P :: u : !A} \qquad \dfrac{\dfrac{\Gamma, u : A; \Delta, x : A \vdash Q :: z : C}{\Gamma, u : A; \Delta \vdash \overline{u}[x].Q :: z : C}}{\Gamma; \Delta, u : !A \vdash \overline{u}[x].Q :: z : C}}{\Gamma; \Delta \vdash (\nu u)(!u(x).P \mid \overline{u}[x].Q :: z : C)}$$

$$\longrightarrow$$

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$$\longrightarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma;\ \emptyset \vdash P :: x : A}{\Gamma, u : A;\ \emptyset \vdash P :: x : A} \qquad \Gamma, u : A;\ \Delta, x : A \vdash Q :: z : C}{\Gamma, u : A;\ \Delta \vdash (\nu x)(P \mid Q) :: z : C}}{\Gamma;\ \Delta, u : !A \vdash (\nu x)(P \mid Q) :: z : C} \qquad \Gamma;\ \emptyset \vdash !u(x).P :: u : !A}{\Gamma;\ \Delta \vdash (\nu u)(!u(x).P \mid (\nu x)(P \mid Q)) :: z : C}$$

Client-Server interaction

$$(\nu u)\big(!u(y).P \mid \overline{u}[z].Q)\big) \longrightarrow (\nu u)\big(!u(y).P \mid (\nu z)(P\{^y/_z\} \mid Q)\big)$$

Client-Server interaction

$$(\nu u)\big(!u(y).P \mid \overline{u}[z].Q\big) \longrightarrow (\nu u)\big(!u(y).P \mid (\nu z)(P\{y/z\} \mid Q)\big)$$

For convenience, we will also use the derived composition rule for servers

$$\frac{\Gamma;\ \Delta_1 \vdash P :: x : !A \qquad \Gamma, x : A;\ \Delta_2 \vdash Q :: z : C}{\Gamma;\ \Delta_1, \Delta_2 \vdash (\nu x)(P \mid Q) :: z : C}\ \text{!-CUT}$$

The πDILL system

!R
$$\frac{\Gamma;\ \emptyset \vdash P :: y : A}{\Gamma;\ \emptyset \vdash\ !u(y).P :: u : !A}$$

!L
$$\frac{u : A, \Gamma;\ y : A, \Delta \vdash Q :: z : C}{u : A, \Gamma;\ \Delta \vdash (\nu y)(\overline{u}\langle y\rangle.Q) :: z : C}$$

!-MOVE
$$\frac{\Gamma;\ \Delta, x : !A \vdash P :: z : C}{u : A, \Gamma;\ \Delta \vdash P\{^{u}/_{x}\} :: z : C}$$

!-CUT
$$\frac{\Gamma;\ \Delta_1 \vdash P :: x : !A \qquad \Gamma, x : A;\ \Delta_2 \vdash Q :: z : C}{\Gamma;\ \Delta_1, \Delta_2 \vdash (\nu x)(P \mid Q) :: z : C}$$

$$(\nu u)\big(!u(y).P \mid (\nu y)(\overline{u}\langle y\rangle \mid Q)\big) \longrightarrow (\nu u)\big(!u(y).P \mid (\nu y)(P \mid Q)\big)$$

Bar Tending at the Linear Logic Cafe

$$\dfrac{\emptyset;\ e : E \otimes E \otimes E \vdash \mathit{deposit}(e, x) :: x : \mathbf{1} \qquad \dfrac{\dfrac{\emptyset;\ k : \mathit{Keg} \vdash \mathit{pour}(k, b) :: b : \mathit{Beer}}{\qquad}}{\qquad}}{\dfrac{\qquad}{\qquad}}$$

Bar Tending at the Linear Logic Cafe

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$$

Bar Tending at the Linear Logic Cafe

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Bar Tending at the Linear Logic Cafe

$$
\dfrac{\emptyset;\ e : E \otimes E \otimes E \vdash deposit(e,x) :: x : \mathbf{1} \qquad \dfrac{\dfrac{\emptyset;\ k : Keg \vdash pour(k,b) :: b : Beer}{u : Keg;\ \emptyset \vdash \overline{u}[k].pour(k,b) :: b : Beer}}{u : Keg;\ x : \mathbf{1} \vdash x().\overline{u}[k].pour(k,b) :: b : Beer}}{\dfrac{u : Keg;\ e : E^3 \vdash (\nu x)\big(deposit(e,x) \mid x().\overline{u}[k].pour(k,b)\big) :: b : Beer}{\dfrac{u : Keg;\ \emptyset \vdash b(e).(\nu x)\big(deposit(e,x) \mid x().\overline{u}[k].pour(k,b)\big) :: b : E^3 \multimap Beer}{u : Keg;\ \emptyset \vdash !t(b).b(e).(\nu x)\big(deposit(e,x) \mid x().\overline{u}[k].pour(k,b)\big) :: t : !(E^3 \multimap Beer)}}}
$$

List-like specification for Servers

$$\dfrac{\dfrac{}{u : A; \emptyset \vdash \ldots :: w : \mathbf{1} \mathbin{\&} (A \otimes {!A})}}{\emptyset; u : {!A} \vdash \ldots :: w : \mathbf{1} \mathbin{\&} (A \otimes {!A})}$$

List-like specification for Servers

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List-like specification for Servers

$$
\dfrac{\dfrac{\dfrac{u : A; \emptyset \vdash {\color{red}\overline{w}\langle\rangle} :: w : \mathbf{1} \qquad \dfrac{}{u : A; \emptyset \vdash \ldots :: w : A \otimes !A}}{u : A; \emptyset \vdash w \triangleright \left\{ \begin{array}{l} \mathtt{inl} : \overline{w}\langle\rangle, \\ \mathtt{inr} : \ldots \end{array} \right\} :: w : \mathbf{1} \,\&\, (A \otimes !A)}}{u : A; \emptyset \vdash w \triangleright \left\{ \begin{array}{l} \mathtt{inl} : \overline{w}\langle\rangle, \\ \mathtt{inr} : \ldots \end{array} \right\} :: w : \mathbf{1} \,\&\, (A \otimes !A)}}{\emptyset; u : !A \vdash w \triangleright \left\{ \begin{array}{l} \mathtt{inl} : \overline{w}\langle\rangle, \\ \mathtt{inr} : \ldots \end{array} \right\} :: w : \mathbf{1} \,\&\, (A \otimes !A)}
$$

List-like specification for Servers

$$\dfrac{u : A;\, \emptyset \vdash \overline{w}\langle\rangle :: w : \mathbf{1} \qquad \dfrac{\begin{array}{c} u : A;\, \emptyset \vdash \color{red}{\overline{u}[x].[y \leftarrow x]} :: y : A \\ u : A;\, \emptyset \vdash \color{red}{[w \leftarrow u]} :: w :: !A \end{array}}{u : A;\, \emptyset \vdash \overline{w}[y](\overline{u}[x].[y \leftarrow x]) \mid [w \leftarrow u]) :: w : A \otimes !A}}{\dfrac{u : A;\, \emptyset \vdash w \triangleright \left\{ \begin{array}{l} \texttt{inl} : \overline{w}\langle\rangle, \\ \texttt{inr} : \ldots \end{array} \right\} :: w : \mathbf{1} \,\&\, (A \otimes !A)}{\dfrac{u : A;\, \emptyset \vdash w \triangleright \left\{ \begin{array}{l} \texttt{inl} : \overline{w}\langle\rangle, \\ \texttt{inr} : \ldots \end{array} \right\} :: w : \mathbf{1} \,\&\, (A \otimes !A)}{\emptyset;\, u : !A \vdash w \triangleright \left\{ \begin{array}{l} \texttt{inl} : \overline{w}\langle\rangle, \\ \texttt{inr} : \ldots \end{array} \right\} :: w : \mathbf{1} \,\&\, (A \otimes !A)}}}$$

Bonus: from IL to ILL

There is a translation $[\![-]\!]$ from intuitionistic logic to intuitionistic linear logic:

$$[\![A \to B]\!] = (![\![A]\!]) \multimap [\![B]\!] \qquad [\![\top]\!] = !\mathbf{1}$$

$$[\![A \wedge B]\!] = [\![A]\!] \mathbin{\&} [\![B]\!] \qquad [\![A \vee B]\!] = ![\![A]\!] \oplus ![\![B]\!]$$

Such that
$\Gamma \vdash_{\mathrm{IL}} A \implies [\![\Gamma]\!]; \emptyset \vdash_{\mathrm{DILL}} [\![A]\!]$