Propositions as Sessions

Logical Foundations of Concurrent Computation

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(Part 1, v1.1)

Linear Logic

Linear Logic

- Interest/applications in proof theory and computer science
- A “resource-aware” logic:
  - $A \vdash B$: given A-resources, a way to obtain B-resources
  - Proof theoretically: substructural logic
- In this course: intuitionistic variant ILL

Linear Logic: Propositions

A is true vs I have A-resources

$A \wedge B$	both A and B are true
$A \to B$	if A is true, then B is true

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$A \multimap B$	if you give me A, then I can produce B

Linear Logic: Propositions

A is true vs I have A-resources

$A \wedge B$	both A and B are true
$A \rightarrow B$	if A is true, then B is true
$A \otimes B$	I have both A and B
$A \multimap B$	if you give me A, then I can produce B

E.g. $euro \otimes euro \multimap pizza$, $dough \otimes (sauce \otimes toppings \multimap pizza)$

Linear Logic: Linearity

$$A \rightarrow A \wedge A \qquad A \not\multimap A \otimes A$$

$$A \wedge B \rightarrow A \qquad A \otimes B \not\multimap A$$

Linear Logic: Linearity

$$A \to A \wedge A \qquad\qquad A \not\multimap A \otimes A$$

$$A \wedge B \to A \qquad\qquad A \otimes B \not\multimap A$$

$$A \wedge (A \to B) \to B \qquad\qquad A \otimes (A \multimap B) \multimap B$$

Linear Logic: Linearity

$$A \to A \wedge A \qquad\qquad A \not\multimap A \otimes A$$

$$A \wedge B \to A \qquad\qquad A \otimes B \not\multimap A$$

$$A \wedge (A \to B) \to B \qquad\qquad A \otimes (A \multimap B) \multimap B$$

$$A \wedge (A \to B) \to A \wedge B \qquad\qquad A \otimes (A \multimap B) \not\multimap A \otimes B$$

Linear Logic is *substructural*, resource-aware.

Proof theory of $\wedge$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A$$

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}\ \wedge\mathrm{R}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge\mathrm{L}$$

$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}\ \textsc{Weakn}$$

$$\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\ \textsc{Contr}$$

$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\ \textsc{eXchg}$$

Proof theory of $\wedge$

Assertion A holds under the list of hypothesis Γ.
Intuitively, $\bigwedge \Gamma \implies A$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A$$

$$\wedge\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\textsc{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C}$$

$$\textsc{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$\textsc{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

Proof theory of $\wedge$

Axiom rule

Sequent calculus: $\Gamma \vdash A$

Right and left rules for $\wedge$

$$A \vdash A$$

$$\wedge\text{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \qquad \wedge\text{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\textsc{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \textsc{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \qquad \textsc{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

Proof theory of $\wedge$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}\ \wedge\text{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge\text{L}$$

$$\underbrace{\frac{\Gamma \vdash C}{\Gamma, A \vdash C}\ \textsc{Weakn} \qquad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\ \textsc{Contr} \qquad \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\ \textsc{eXchg}}_{\boxed{\text{Structural rules}}}$$

Proof theory of $\wedge$

$\Gamma : \mathit{Frml} \to \mathbb{N}$ is a multiset

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}\,\wedge\mathrm{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge\mathrm{L}$$

$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}\,\textsc{Weakn} \qquad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\,\textsc{Contr} \qquad \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\,\textsc{eXchg}$$

We can omit the exchange rule by treating Γ as a multiset

Generalized structural rules

$$\frac{\Gamma, \Gamma \vdash C}{\Gamma \vdash C} \qquad \frac{\Gamma \vdash C}{\Gamma, \Delta \vdash C}$$

Obtained from repeatedly using the usual rules (and exchange).

Using structural rules

$$\frac{}{A \wedge B \vdash A}$$

$$\frac{}{A \vdash A \wedge A}$$

Using structural rules

$$\dfrac{\overline{A, B \vdash A}}{A \wedge B \vdash A} \qquad\qquad \dfrac{}{A \vdash A \wedge A}$$

Using structural rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A}$$

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Using structural rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A} \qquad\qquad \dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A, A \vdash A \wedge A}}{A \vdash A \wedge A}$$

Using structural rules

$$\dfrac{\dfrac{A \vdash A}{A, B \vdash A}}{A \wedge B \vdash A} \qquad\qquad \dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A, A \vdash A \wedge A}}{A \vdash A \wedge A}$$

Crucial is the usage of the contraction and weakening rules

Associativity

$$\frac{}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}$$

Associativity

$$\dfrac{\dfrac{}{A, B, C \vdash (A \wedge B) \wedge C}}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}$$

Associativity

$$\dfrac{\dfrac{}{A, B \vdash A \wedge B} \qquad C \vdash C}{\dfrac{A, B, C \vdash (A \wedge B) \wedge C}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}}$$

Associativity

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B} \qquad C \vdash C}{\dfrac{A, B, C \vdash (A \wedge B) \wedge C}{A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C}}$$

Alternative rules for $\wedge$

We can replace $\wedge L$ with the following two rules:

$$\wedge\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \qquad \longrightarrow \qquad \wedge\mathrm{L1}\ \frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \qquad \wedge\mathrm{L2}\ \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

These rules *internalize weakining*.

$$\frac{}{\Gamma, A \wedge B \vdash C}$$

$$\dfrac{\overline{\Gamma, A \wedge B, A \wedge B \vdash C}}{\Gamma, A \wedge B \vdash C}$$

$$\cfrac{\cfrac{\overline{\Gamma, A, A \wedge B \vdash C}}{\Gamma, A \wedge B, A \wedge B \vdash C}}{\Gamma, A \wedge B \vdash C}$$

$$\dfrac{\dfrac{\dfrac{\Gamma, A, B \vdash C}{\Gamma, A, A \wedge B \vdash C}}{\Gamma, A \wedge B, A \wedge B \vdash C}}{\Gamma, A \wedge B \vdash C}$$

Alternative rules for $\wedge$

We can replace $\wedge R$ with the following rule:

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \qquad \longrightarrow \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

From intuitionistic to linear logic

- Main idea: treat $\otimes$ the same as $\wedge$, minus contraction and weakening rules.
- The context Γ then externalizes $\otimes$ on the meta-level:

$$\Gamma \vdash A \qquad \longrightarrow \qquad \bigotimes \Gamma \implies A$$

- We will recover contraction and weakening later with the ! modality.

Proof theory of $\otimes$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A$$

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \otimes\mathrm{R}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes\mathrm{L}$$

$$\cancel{\frac{\Gamma \vdash C}{\Gamma, A \vdash C}\ \textsc{Weakn}}$$

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$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\ \textsc{eXchg}$$

Proof theory of $\otimes$

Left and right rules for $\otimes$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A \qquad \otimes\text{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \qquad \otimes\text{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\cancel{\textsc{Weakn}\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C}} \qquad \cancel{\textsc{Contr}\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}} \qquad \textsc{eXchg}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}$$

Proof theory of $\otimes$

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\,{\otimes}\mathrm{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,{\otimes}\mathrm{L}$$

$$\cancel{\frac{\Gamma \vdash C}{\Gamma, A \vdash C}\,\textsc{Weakn}} \qquad \cancel{\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}\,\textsc{Contr}} \qquad \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}\,\textsc{eXchg}$$

No weakening or contraction

Adding the unit

A proof theory with just $\otimes$ is not fun. We need multiplicative unit ($\mathbf{1}$) to signify absence of resouces:

$$\emptyset \vdash \mathbf{1} \qquad\qquad \frac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}$$

Adding implication

A proof theory with just $\otimes$, $\mathbf{1}$ is not fun. We need linear implication ($\multimap$, *lollipop*) to form linear hypotheticals:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$$

Adding implication

A proof theory with just $\otimes$, $\mathbf{1}$ is not fun. We need linear implication ($\multimap$, *lollipop*) to form linear hypotheticals:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad\qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$$

Exercise: try to derive $(A \otimes B) \multimap C \dashv\vdash A \multimap (B \multimap C)$

Cut rule

$$\dfrac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\ \textsc{Cut}$$

What is so special about this rule?
Each cut rule introduces a piece of "complexity" as a new formula A

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Each cut rule introduces a piece of "complexity" as a new formula A

Theorem

Every proof of $\Gamma \vdash A$ that uses cut can be converted into a proof that does not use cut

MILL

$$A, B \quad ::= \quad \mathbf{1} \mid A \otimes B \mid A \multimap B$$

$$A \vdash A \qquad \emptyset \vdash \mathbf{1} \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \otimes\mathrm{R} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes\mathrm{L}$$

$$\frac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B}\ \textsc{Cut} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap\mathrm{R} \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}\ \multimap\mathrm{L}$$

Adding &

Recall the different versions of left/right rules for $\wedge$

$$\wedge\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

$$\wedge\mathrm{R'}\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\wedge\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$\wedge\mathrm{L}_i\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}$$

Adding &

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$$\otimes\mathrm{R}\quad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}$$

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$$\otimes\mathrm{L}\quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\wedge\mathrm{L}_i\quad \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}$$

Adding &

Recall the different versions of left/right rules for $\wedge$

$$\frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \otimes\mathrm{R}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \mathbin{\&} B}\ \&\mathrm{R}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes\mathrm{L}$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \mathbin{\&} A_2 \vdash C}\ \&\mathrm{L}_i$$

Interplay of $\otimes$ and &

- $A \mathbin{\&} B \vdash A$, and $A \mathbin{\&} B \vdash B$
- $A \mathbin{\&} B \not\dashv\vdash A \otimes B$
- $(A \multimap B) \mathbin{\&} (A \multimap C) \vdash A \multimap B \mathbin{\&} C$
- $(A \mathbin{\&} B) \multimap C \nvdash A \multimap (B \multimap C)$

Menu at the Linear Logic Cafe

Summer Menu! €15 p.p.
Starter: Greek Salad, or Soup
Main: Chicken Schnitzel, Tofu, or Salmon (for additional €2)
(all main dishes come with a side of fries)
Desert: Ice Cream, or Cheese Platter
Drinks: Leffe Tripel (for additional €3)

$$E^{15} \multimap \left(\qquad \right)$$

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$$E^{15} \multimap \left((Sal \mathbin{\&} Soup) \otimes \right)$$

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$$E^{15} \multimap \left(\begin{array}{l} (Sal \mathbin{\&} Soup) \otimes \\ ((Schn \mathbin{\&} Tofu \mathbin{\&} (E \otimes E \multimap Fish)) \\ \\ \\ \\ \end{array} \right)$$

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$$
\begin{gathered}
E^{15} \multimap \\
\left(\begin{array}{l}
(Sal \;\&\; Soup) \otimes \\
((Schn \;\&\; Tofu \;\&\; (E \otimes E \multimap Fish)) \\
\qquad \otimes \;\; Fries) \otimes \\
(Icecr \;\&\; Cheese) \otimes \\
\\
\\
\end{array}\right)
\end{gathered}
$$

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E^{15} \multimap \left(\begin{array}{l}
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((Schn \mathbin{\&} Tofu \mathbin{\&} (E \otimes E \multimap Fish)) \\
\qquad \otimes \; Fries) \otimes \\
(Icecr \mathbin{\&} Cheese) \otimes \\
\\
(E^3 \multimap Beer)
\end{array}\right)
$$

Grandma's Linear Logic Pizza recipe

LL Pizza Ingredients:

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

$$\left(\qquad\qquad \right) \multimap \mathit{Pizza}$$

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$$\left(\begin{array}{l} \mathit{Yeast} \otimes \mathit{Flour} \otimes \\ \\ \\ \\ \end{array}\right) \multimap \mathit{Pizza}$$

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$$\left(\begin{array}{l} \mathit{Yeast} \otimes \mathit{Flour} \otimes \\[1em] \mathit{Salt} \otimes \mathit{Sugar} \otimes \\[4em] \end{array}\right) \multimap \mathit{Pizza}$$

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$$\begin{pmatrix} \mathit{Yeast} \otimes \mathit{Flour} \otimes \\ \mathit{Salt} \otimes \mathit{Sugar} \otimes \\ \mathit{Tomatoes} \otimes \\ \\ \end{pmatrix} \multimap \mathit{Pizza}$$

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$$\left(\begin{array}{l} \mathit{Yeast} \otimes \mathit{Flour} \otimes \\[1em] \mathit{Salt} \otimes \mathit{Sugar} \otimes \\[1em] \mathit{Tomatoes} \otimes \\[1em] (\mathit{Sausage} \text{ or } \mathit{Paprika}) \end{array}\right) \multimap \mathit{Pizza}$$

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$$\left(\begin{array}{l} \mathit{Yeast} \otimes \mathit{Flour} \otimes \\ \mathit{Salt} \otimes \mathit{Sugar} \otimes \\ \mathit{Tomatoes} \otimes \\ (\mathit{Sausage} \oplus \mathit{Paprika}) \end{array}\right) \multimap \mathit{Pizza}$$

Adding $\oplus$

$$\oplus R_1 \quad \frac{\Gamma \vdash A_1}{\Gamma \vdash A_1 \oplus A_2} \qquad\qquad \oplus R_2 \quad \frac{\Gamma \vdash A_2}{\Gamma \vdash A_1 \oplus A_2}$$

$$\oplus L \quad \frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 \oplus A_2 \vdash C}$$

$\oplus$ vs $\&$

The rules are the same as for $\vee$ in intuitionistic sequent calculus.

$\oplus$ vs $\&$

The rules are the same as for $\vee$ in intuitionistic sequent calculus.

However,

$$A \,\&\, (B \oplus C) \nvdash (A \,\&\, B) \oplus (A \,\&\, C)$$

- $A =$ one euro;
- $B =$ tea, $C =$ coffee;
- $(B \oplus C) =$ either tea or coffee, but do not know which;
- $A \,\&\, (B \oplus C) =$ i can either have a euro or a beverage

$\oplus$ vs $\&$

$$\mathit{tea} \oplus \mathit{coffee} \vdash \mathit{awake}$$

$$\mathit{tea} \mathbin{\&} \mathit{coffee} \vdash \mathit{awake}$$

$\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

$\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

$\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

$tea \,\&\, coffee \vdash tea \oplus coffee$

$\oplus$ vs $\&$

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

$tea \,\&\, coffee \vdash awake$

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

$tea \,\&\, coffee \vdash tea \oplus coffee$

$A \,\&\, B \vdash A \oplus B \qquad A \oplus B \nvdash A \,\&\, B$

MAILL

$$A, B \quad ::= \quad \mathbf{1} \mid A \otimes B \mid A \multimap B \mid A \mathbin{\&} B \mid A \oplus B$$

$$A \vdash A \qquad\qquad \emptyset \vdash \mathbf{1} \qquad\qquad \otimes\mathrm{R}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B} \qquad\qquad \otimes\mathrm{L}\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\textsc{Cut}\ \frac{\Gamma \vdash A \qquad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B} \qquad \multimap\mathrm{R}\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \multimap\mathrm{L}\ \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C} \qquad \&\mathrm{R}\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \mathbin{\&} B}$$

$$\&\mathrm{L}_i\ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \mathbin{\&} A_2 \vdash C} \qquad \oplus\mathrm{R}_i\ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \oplus A_2} \qquad \oplus\mathrm{L}\ \frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 \oplus A_2 \vdash C}$$

Related Logics

- Classical Linear Logic

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- Categorical logics
  - Like MILL but without the exchange rule
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Related Logics

- Classical Linear Logic
- Categorical logics
  - Like MILL but without the exchange rule
  - Loose commutativity of $\otimes$
  - Two implications $A \backslash B$ and B / A
  - Uses in linguistics
- Logic of Bunched Implications
  - Better known as the kernel of Separation Logic
  - Popular in deductive program verification
- More on variants in the Linear Logic course next week

Upcoming

- Tomorrow and Wednesday: more on &, $\otimes$, $\multimap$ and also $\oplus$; their computational interpretaion.
- Thursday: ! modality and its computational interpretation
- Friday: computational interpretation of BI