Propositions as Sessions Logical Foundations of Concurrent Computation

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> ESSLLI 2024 (Part 1, v1.1)

Linear Logic

Linear Logic

- Interest/applications in proof theory and computer science
- A "resource-aware" logic:
 - A \vdash B: given A-resources, a way to obtain B-resources
 - Proof theoretically: substructural logic
- In this course: intuitionistic variant ILL

A is true vs I have A-resources

$$A \land B$$
both A and B are true $A \rightarrow B$ if A is true, then B is true

A is true vs I have A-resources

 $A \land B$ both A and B are true $A \rightarrow B$ if A is true, then B is true

$A \otimes B$ I have both A and B

A is true vs I have A-resources

$oldsymbol{A}\wedgeoldsymbol{B}$	both A and B are true
$oldsymbol{A} ightarrow oldsymbol{B}$	if A is true, then B is true

$A \otimes B$ I have both A and B $A \longrightarrow B$ if you give me A, then I can produce B

A is true vs I have A-resources

 $A \land B$ both A and B are true $A \rightarrow B$ if A is true, then B is true

 $A \otimes B$ I have both A and B $A \multimap B$ if you give me A, then I can produce B

E.g. $euro \otimes euro \longrightarrow pizza$, $dough \otimes (sauce \otimes toppings \longrightarrow pizza)$

Linear Logic: Linearity

$$A \rightarrow A \land A$$
 $A \not\rightarrow a \otimes A$ $A \land B \rightarrow A$ $A \otimes B \not\rightarrow a$

Linear Logic: Linearity

$$\begin{array}{ccc} A \to A \land A & A \not \sim A \otimes A \\ A \land B \to A & A \otimes B \not \sim A \\ A \land (A \to B) \to B & A \otimes (A \multimap B) \multimap B \end{array}$$

Linear Logic: Linearity

$$\begin{array}{ccc} A \to A \land A & A \not \sim A \otimes A \\ A \land B \to A & A \otimes B \not \sim A \\ A \land (A \to B) \to B & A \otimes (A \multimap B) \multimap B \\ A \land (A \to B) \to A \land B & A \otimes (A \multimap B) \not \sim A \otimes B \end{array}$$

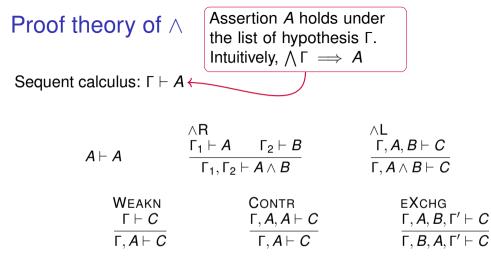
Linear Logic is *substructural*, resource-aware.

Proof theory of \wedge

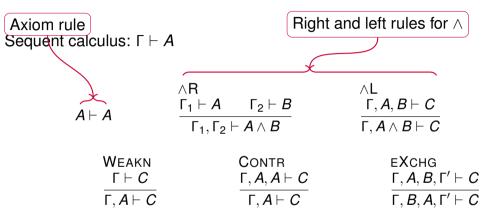
Sequent calculus: $\Gamma \vdash A$

$$A \vdash A \qquad \begin{array}{c} \wedge \mathbf{R} & & \wedge \mathbf{L} \\ \frac{\Gamma_1 \vdash A}{\Gamma_1, \Gamma_2 \vdash A \wedge B} & & \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \\ \end{array}$$

$$\begin{array}{c} \mathsf{WEAKN} & & \mathsf{CONTR} \\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} & & \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} & & \frac{\mathsf{EXCHG}}{\Gamma, B, A, \Gamma' \vdash C} \end{array}$$

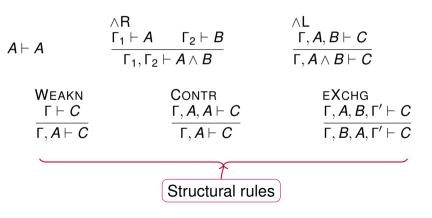


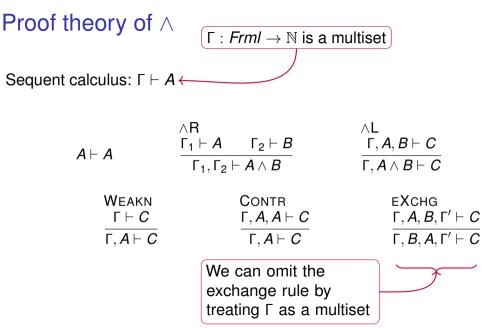
Proof theory of \wedge



Proof theory of \wedge

Sequent calculus: $\Gamma \vdash A$





Generalized structural rules



Obtained from repeatedly using the usual rules (and exchange).

July 28, 2024

$\overline{A \land B \vdash A}$

 $A \vdash A \land A$

 $\frac{\overline{A, B \vdash A}}{\overline{A \land B \vdash A}}$

 $A \vdash A \land A$

$$\frac{A \vdash A}{\overline{A, B \vdash A}}$$

$$\overline{A \land B \vdash A}$$

 $A \vdash A \land A$

$$\frac{A \vdash A}{\overline{A, B \vdash A}}$$

$$\overline{A \land B \vdash A}$$

$$\frac{A \vdash A}{\overline{A, B \vdash A}}$$

 $\frac{A \vdash A \qquad A \vdash A}{A, A \vdash A \land A}$ $\frac{A \vdash A \land A}{A \vdash A \land A}$



Crucial is the usage of the contraction and weakening rules

July 28, 2024

 $A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C$

Associativity

Associativity

$\overline{ \begin{array}{c} \textbf{A}, \textbf{B}, \textbf{C} \vdash (\textbf{A} \land \textbf{B}) \land \textbf{C} \\ \textbf{A} \land (\textbf{B} \land \textbf{C}) \vdash (\textbf{A} \land \textbf{B}) \land \textbf{C} \end{array} }$

Associativity

$oldsymbol{A},oldsymbol{B}dasholdsymbol{A}\wedgeoldsymbol{B}$	$C \vdash C$
$A, B, C \vdash (A \land$	$(B) \land C$
$A \wedge (B \wedge C) \vdash (A \wedge C)$	$(A \land B) \land C$

Associativity

$A \vdash A$	$B \vdash B$	
<i>A</i> , <i>B</i> ⊢	$A \wedge B$	$\mathcal{C} \vdash \mathcal{C}$
$egin{aligned} egin{aligned} egin{aligned} egin{aligned} eta, B, C dash (A \wedge B) \wedge C \end{aligned} \end{aligned}$		
$A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C$		

Alternative rules for \wedge

We can replace $\wedge L$ with the following two rules:

$$\begin{array}{ccc} \wedge \mathsf{L} & & \wedge \mathsf{L1} & & \wedge \mathsf{L2} \\ \hline \Gamma, \mathsf{A} \wedge \mathsf{B} \vdash \mathsf{C} & & & \hline \Gamma, \mathsf{A} \wedge \mathsf{B} \vdash \mathsf{C} & & \hline \Gamma, \mathsf{A} \wedge \mathsf{B} \vdash \mathsf{C} & & \hline \Gamma, \mathsf{A} \wedge \mathsf{B} \vdash \mathsf{C} & & \hline \Gamma, \mathsf{A} \wedge \mathsf{B} \vdash \mathsf{C} & & \hline \end{array}$$

These rules internalize weakining.

$\Gamma, A \land B \vdash C$

$\frac{\overline{\Gamma, \boldsymbol{A} \land \boldsymbol{B}, \boldsymbol{A} \land \boldsymbol{B} \vdash \boldsymbol{C}}}{\Gamma, \boldsymbol{A} \land \boldsymbol{B} \vdash \boldsymbol{C}}$

 $\frac{\overline{\Gamma, A, A \land B \vdash C}}{\overline{\Gamma, A \land B, A \land B \vdash C}}{\overline{\Gamma, A \land B \vdash C}}$

 $\frac{ \begin{matrix} \mathsf{\Gamma}, \mathsf{A}, \mathsf{B} \vdash \mathsf{C} \\ \hline \mathsf{\Gamma}, \mathsf{A}, \mathsf{A} \land \mathsf{B} \vdash \mathsf{C} \\ \hline \\ \hline \mathsf{\Gamma}, \mathsf{A} \land \mathsf{B}, \mathsf{A} \land \mathsf{B} \vdash \mathsf{C} \\ \hline \\ \hline \mathsf{\Gamma}, \mathsf{A} \land \mathsf{B} \vdash \mathsf{C} \end{matrix}$

Alternative rules for \wedge

We can replace $\wedge R$ with the following rule:

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \land B} \longrightarrow \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B}$$

From intuitionistic to linear logic

- Main idea: treat \otimes the same as \wedge , minus contraction and weakening rules.
- The context Γ then externalizes \otimes on the meta-level:

$$\Gamma \vdash A \longrightarrow \bigotimes \Gamma \Longrightarrow A$$

▶ We will recover contraction and weakening later with the ! modality.

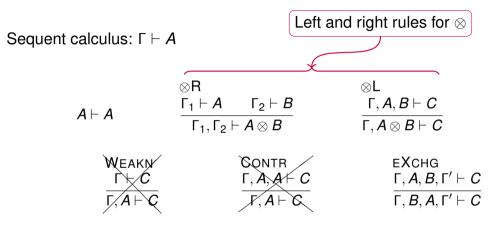
Proof theory of \otimes

Sequent calculus: $\Gamma \vdash A$

$$A \vdash A \qquad \begin{array}{c} \otimes \mathsf{R} \\ \Gamma_1 \vdash A & \Gamma_2 \vdash B \\ \hline \Gamma_1, \Gamma_2 \vdash A \otimes B \end{array} \qquad \begin{array}{c} \otimes \mathsf{L} \\ \hline \Gamma, A, B \vdash C \\ \hline \Gamma, A \otimes B \vdash C \end{array}$$

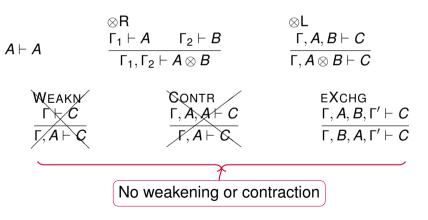
$$WEAKN \qquad \begin{array}{c} \otimes \mathsf{CONTR} \\ \hline \Gamma, A, A \vdash C \\ \hline \Gamma, A \vdash C \end{array} \qquad \begin{array}{c} \mathsf{EXCHG} \\ \hline \Gamma, A, B, \Gamma' \vdash C \\ \hline \Gamma, B, A, \Gamma' \vdash C \end{array}$$

Proof theory of \otimes



Proof theory of \otimes

Sequent calculus: $\Gamma \vdash A$



A proof theory with just \otimes is not fun. We need multiplicative unit (1) to signify absence of resouces:

$$\emptyset \vdash \mathbf{1} \qquad \qquad \frac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}$$

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Adding implication

A proof theory with just \otimes , **1** is not fun. We need linear implication ($-\infty$, *lollipop*) to form linear hypotheticals:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \qquad \overline{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$$

Adding implication

A proof theory with just \otimes , **1** is not fun. We need linear implication ($-\infty$, *lollipop*) to form linear hypotheticals:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$$

Exercise: try to derive $(A \otimes B) \multimap C \dashv A \multimap (B \multimap C)$

Cut rule

$$\frac{\begin{array}{c} \mathsf{Cut} \\ \Gamma \vdash \mathbf{A} & \Gamma', \mathbf{A} \vdash \mathbf{B} \\ \hline \Gamma, \Gamma' \vdash \mathbf{B} \end{array}$$

What is so special about this rule?

Each cut rule introduces a piece of "complexity" as a new formula A

Cut rule

 $\frac{\begin{array}{c} \mathsf{Cut} \\ \Gamma \vdash \mathbf{A} & \Gamma', \mathbf{A} \vdash \mathbf{B} \\ \hline \Gamma, \Gamma' \vdash \mathbf{B} \end{array}$

What is so special about this rule?

Each cut rule introduces a piece of "complexity" as a new formula A

Theorem

Every proof of $\Gamma \vdash A$ that uses cut can be converted into a proof that does not use cut

MILL

 $A, B ::= \mathbf{1} | A \otimes B | A \multimap B$ $\otimes \mathbf{R}$ $\otimes \mathsf{L}$ $\Gamma_1 \vdash A$ $\Gamma_2 \vdash B$ Γ, *A*, *B* ⊢ *C* $A \vdash A$ Ø⊢1 $\overline{\Gamma_1,\Gamma_2\vdash A\otimes B} \qquad \overline{\Gamma,A\otimes B\vdash C}$ Сит $- \mathbf{R}$ _₀L $\Gamma \vdash A \qquad \Gamma', A \vdash B$ $\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \multimap B \vdash C}$ Γ.Γ' ⊢ *B*



Recall the different versions of left/right rules for \wedge

$$\begin{array}{l} \wedge \mathbf{R} \\ \Gamma_{1} \vdash \mathbf{A} \quad \Gamma_{2} \vdash \mathbf{B} \\ \hline \Gamma_{1}, \Gamma_{2} \vdash \mathbf{A} \wedge \mathbf{B} \end{array} \qquad \qquad \begin{array}{l} \wedge \mathbf{R}^{\prime} \\ \Gamma \vdash \mathbf{A} \quad \Gamma \vdash \mathbf{B} \\ \hline \Gamma \vdash \mathbf{A} \wedge \mathbf{B} \end{array}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \land B \vdash C}$$

 $\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \land A_2 \vdash C}$



Recall the different versions of left/right rules for \wedge

$$\begin{array}{c} \otimes \mathsf{R} \\ \hline \Gamma_1 \vdash A & \Gamma_2 \vdash B \\ \hline \Gamma_1, \Gamma_2 \vdash A \otimes B \end{array} & \begin{array}{c} \wedge \mathsf{R}' \\ \hline \Gamma \vdash A & \Gamma \vdash B \\ \hline \Gamma \vdash A \wedge B \end{array} \\ \\ \otimes \mathsf{L} \\ \hline \Gamma, A, B \vdash C \\ \hline \Gamma, A \otimes B \vdash C \end{array} & \begin{array}{c} \wedge \mathsf{L}_i \\ \hline \Gamma, A_i \vdash C \\ \hline \Gamma, A_1 \wedge A_2 \vdash C \end{array}$$



Recall the different versions of left/right rules for \wedge

$$\begin{array}{c} \otimes \mathsf{R} \\ \hline \Gamma_1 \vdash A & \Gamma_2 \vdash B \\ \hline \Gamma_1, \Gamma_2 \vdash A \otimes B \end{array} & \begin{array}{c} \& \mathsf{R} \\ \hline \Gamma \vdash A & \Gamma \vdash B \\ \hline \Gamma \vdash A \& B \end{array} \\ \\ \end{array} \\ \\ \end{array} \\ \\ \begin{array}{c} \otimes \mathsf{L} \\ \hline \Gamma, A, B \vdash C \\ \hline \Gamma, A \otimes B \vdash C \end{array} & \begin{array}{c} \& \mathsf{L}_i \\ \hline \Gamma, A_i \vdash C \\ \hline \Gamma, A_1 \& A_2 \vdash C \end{array} \end{array}$$

Interplay of \otimes and &

- $\blacktriangleright A \& B \vdash A, \text{ and } A \& B \vdash B$
- $\blacktriangleright A \& B \not \vdash A \otimes B$
- $\blacktriangleright (A \multimap B) \& (A \multimap C) \vdash A \multimap B \& C$
- $\blacktriangleright (A \& B) \multimap C \not\vdash A \multimap (B \multimap C)$

Summer Menu! €15 p.p. Starter: Greek Salad, or Soup Main: Chicken Schnitzel. Tofu, or Salmon (for additional €2) (all main dishes come with a side of fries) Desert: Ice Cream, or Cheese Platter Drinks: Leffe Tripel (for additional €3)

 F^{15} ____

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 $\frac{15}{6}$ (Sal & Soup) \otimes

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 $(Sal \& Soup) \otimes$ $((Schn \& Tofu \& (E \otimes E \multimap Fish))$

Summer Menu! €15 p.p. Starter: Greek Salad, or Soup Main: Chicken Schnitzel. Tofu, or Salmon (for additional €2) (all main dishes come with a side of fries) Desert: Ice Cream, or Cheese Platter Drinks: Leffe Tripel (for additional €3)

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\widetilde{(Sal \& Soup)} \otimes((Schn \& Tofu \& (E \otimes E \multimap Fish))
             \otimes Fries) \otimes
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Summer Menu! €15 p.p. Starter: Greek Salad, or Soup Main: Chicken Schnitzel. Tofu, or Salmon (for additional €2) (all main dishes come with a side of fries) Desert: Ice Cream, or Cheese Platter Drinks: Leffe Tripel (for additional €3)

```
(Sal \& Soup) \otimes
((Schn \& Tofu \& (E \otimes E \multimap Fish))
          \otimes Fries) \otimes
 (Icecr & Cheese) ⊗
```

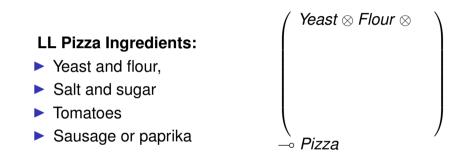
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```
(Sal \& Soup) \otimes
((Schn \& Tofu \& (E \otimes E \multimap Fish))
          \otimes Fries) \otimes
 (Icecr & Cheese) ⊗
 (E^3 \longrightarrow Beer)
```

LL Pizza Ingredients:

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

⊸ Pizza



LL Pizza Ingredients:

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

Yeast ⊗ Flour ⊗
 Salt ⊗ Sugar ⊗
 Pizza

LL Pizza Ingredients:

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

 $\begin{pmatrix}
\text{Yeast} \otimes \text{Flour} \otimes \\
\text{Salt} \otimes \text{Sugar} \otimes \\
\text{Tomatoes} \otimes \\
- & \text{Pizza}
\end{pmatrix}$

LL Pizza Ingredients:

- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

Yeast \otimes Flour \otimes

 $Salt\otimes Sugar\otimes$

 $\mathit{Tomatoes} \otimes$

LL Pizza Ingredients:

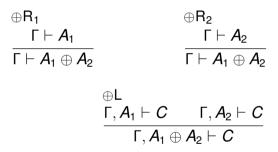
- Yeast and flour,
- Salt and sugar
- Tomatoes
- Sausage or paprika

Yeast \otimes Flour \otimes

 $Salt\otimes Sugar\otimes$

 $\mathit{Tomatoes} \otimes$

(Sausage ⊕ Paprika) , ⊸ Pizza Adding \oplus





The rules are the same as for \lor in intuitionistic sequent calculus.



The rules are the same as for \vee in intuitionistic sequent calculus.

However,

$A \And (B \oplus C) \nvDash (A \And B) \oplus (A \And C)$

- A =one euro;
- \blacktriangleright *B* = tea, *C* = coffee;
- $(B \oplus C)$ = either tea or coffee, but do not know which;
- $A \& (B \oplus C) = i$ can either have a euro or a beverage



 $tea \oplus coffee \vdash awake$

tea & coffee ⊢ awake

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

tea & coffee ⊢ awake

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

tea & coffee ⊢ awake

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

tea & coffee ⊢ awake

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

tea & *coffee* \vdash *tea* \oplus *coffee*

$tea \oplus coffee \vdash awake$

Given either tea or coffee (I don't care which one), I can drink it and get awake

tea & coffee ⊢ awake

Given a choice of tea and coffee (for example at a hotel breakfast), I can drink a hot beverage and get awake

 $tea \& coffee \vdash tea \oplus coffee$ $A \& B \vdash A \oplus B \qquad A \oplus B \nvDash A \& B$

MAILL

$A, B ::= \mathbf{1} | A \otimes B | A \multimap B | A \otimes B | A \oplus B$

$A \vdash A$ $\emptyset \vdash 1$		$\frac{\Gamma_2 \vdash B}{\vdash A \otimes B}$	$\frac{\otimes L}{\Gamma, A, E}$		
$\frac{C U T}{\frac{\Gamma \vdash \boldsymbol{A}}{\Gamma, \Gamma' \vdash \boldsymbol{B}}}$	$\frac{\stackrel{\multimap}{\Gamma} R}{\stackrel{\Gamma}{\Gamma} + A \stackrel{\frown}{\to} B}$	· · ·	$\frac{\Gamma_2, B \vdash C}{A \multimap B \vdash C}$	&R Γ⊢ <i>Α</i> Γ⊢ <i>Α</i>	
$\frac{\&L_i}{\Gamma, \mathcal{A}_i \vdash \mathcal{C}} \\ \frac{\Gamma, \mathcal{A}_i \vdash \mathcal{C}}{\Gamma, \mathcal{A}_1 \And \mathcal{A}_2 \vdash \mathcal{C}}$	$\frac{\oplus R_i}{\Gamma \vdash A_1}$	·	$\frac{\oplus L}{\Gamma, \mathcal{A}_1 \vdash \mathcal{C}}{\Gamma, \mathcal{A}_1 \oplus \mathcal{C}}$, _	

Propositions as Sessions

Classical Linear Logic

- Classical Linear Logic
- Categorical logics
 - Like MILL but without the exchange rule
 - ► Loose commutativity of ⊗

- Classical Linear Logic
- Categorical logics
 - Like MILL but without the exchange rule
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 - Two implications $A \setminus B$ and B/A
 - Uses in linguistics

- Classical Linear Logic
- Categorical logics
 - Like MILL but without the exchange rule
 - ► Loose commutativity of ⊗
 - Two implications $A \setminus B$ and B/A
 - Uses in linguistics
- Logic of Bunched Implications
 - Better known as the kernel of Separation Logic
 - Popular in deductive program verification
- More on variants in the Linear Logic course next week



► Tomorrow and Wednesday: more on &, ⊗, --∞ and also ⊕; their computational interpretaion.

Thursday: ! modality and its computational interpretation

Friday: computational interpretation of BI